An Improved Algorithm for Image Fractal Dimension

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Abstract. There are several methods to estimate fractal dimension in fractional Brownian motion model. In this paper, one of methods, the variance method, is analyzed detailedly, and some problems in the method are pointed out. To resolve the problems, an improved method is proposed. To validate the method, a comparing experiment of two methods is designed. In the experiment, a SPOT image is selected and five categories, flourishing field, bareness land, resident area, water area and mountainous area, in the image are taken as examples. The experiment shows that the result by improved method is coincident with human perception and the improved method is also helpful for classification.

Keywords: fractal dimension, fractional Brownian motion, texture, classification

1. Introduction

Mandelbrot has introduced the concept of fractional Brownian motion (FBM) as a generalization of Brownian motion [1]. Hunt et al. indicate that the statistics of many natural phenomena are indeed best represented as FBM [2]. So, people have expected to describe and classify many natural phenomena in an image with the theory of FBM. An important characteristic of fractals useful for their description and classification is their fractal dimension. Several methods have been developed to estimate fractal dimension in an image. Generally, the methods can be cataloged as variance method, spectral method and morphological method. The variance method is often used in remote sensing classification, but the method of the variance method commonly used in papers has some problems. In this paper, the problems are pointed out and an improved method is proposed.

2. Analysis of the variance method

2.1. The theory of the variance method

Fractional Brownian motion in one dimension $F(X)$ is a single valued function of one variable, usually time. Mandelbrot defines the random function $F(X)$ as follows:

$$B(t) = P\left(\frac{F(X + \Delta X) - F(X)}{\Delta X^H} < t\right)$$

(1)

Where, $B(t)$ is the cumulative distribution function of a zero mean Gaussian random variable, and $H$ is known as Hurst exponent. The fractal dimension $D$ of a fractional Brownian motion is simply related to the Hunt exponent $H$ as

$$D = 2 - H$$

(2)

From Eq.(1), we can get an equivalent representation of fractional Brownian motion as follow

$$F(X + \Delta X) - F(X) = k\xi |\Delta X|^H$$

(3)

where $k$ is a constant and $\xi$ is a normalized independent Gaussian random process. We note that fractional Brownian motion is continuous but nondifferentiable.
A fractional Brownian motion has zero average increments:

$$E[F(X + \Delta X) - F(X)] = 0$$ \hfill (4)

Here \(E\) denotes expectation.

The variance of increments \(V(\Delta X)\) is proportional to \(|\Delta X|^{2H}\)

$$V(\Delta X) = E[(F(X + \Delta X) - F(X))^2] = k^2 |\Delta X|^{2H}$$ \hfill (5)

An important property of fractional Brownian motion is that \((F(X + \Delta X) - F(X))\) and \(r^{-H}(F(X + r\Delta X) - F(X))\) have the same finite dimensional joint distribution functions for any \(\Delta X\) and \(r > 0\), which means that the shape of the two random functions \((F(X + \Delta X) - F(X))\) and \(r^{-H}(F(X + r\Delta X) - F(X))\) are statistically indistinguishable. This property is known as statistical self-affinity.

The variance method is based on the property that the variance of fractal Brownian motion increments is proportional to the time increments to the power 2H. Taking the logarithm of Eq.(5) yields a linearized equation:

$$\log V(\Delta X) = 2H \log |\Delta X| + C$$ \hfill (6)

where \(C\) is a constant. To estimate \(H\), and thus \(D\), the quantities \(V(\Delta X)\) for various \(\Delta X\) are computed, and a least-squares regression on Eq. (6) is used.

2.2. The application of the variance method in digital image

The key of estimating \(D\) is to calculate \((F(X + \Delta X) - F(X))\). If \(F(X)\) denotes an image, then \(X=(x,y)\) represents a pixel and \(F(X)=F(x,y)\) is the value of the pixel \((x,y)\). Because the variance \(X\) is 2-dimensional, the fractal dimension \(D\) of a fractional Brownian motion is as

$$D=3-H$$ \hfill (7)

Usually, we calculate \((F(X + \Delta X) - F(X))\) as follow:

$$|F(X+\Delta X) - F(X)| = \frac{1}{3} \left\{|F(x+\Delta x, y) - F(x, y)| + |F(x, y+\Delta y) - F(x, y)| + |F(x+\Delta x, y+\Delta y) - F(x, y)|\right\}$$ \hfill (8)

Where, \(\triangle X=(\triangle x, \triangle y)\). Selecting a \((\triangle x, \triangle y)\), we can calculate a value of \(E[(F(X+\Delta X)-F(X))^2]\), and thus the value of \(V(\Delta X)\). Changing the values of \(\triangle x\) and \(\triangle y\), we can get a serial of \(V(\Delta X)\), and with Eq.(6) to get \(H\).

2.3. The analyzing of the method

In fact, \(\Delta X\) is a kind of scale. According to the famous problem “the length of coastline”, it is known that the changing of scale will lead to the changing of measure result. Usually, a smaller scale has more power to describe details, and can get a higher resolution. Image is a kind of observing value for the world. When the observing scale is changed, the images should be different, more clearly, to be different resolution (shown in Fig.1). But according to Eq.(8), no matter how to change the scale \(\Delta X\), the image, which used to calculate the \(V(\Delta X)\), is always the \(F(X, y)\) (shown in Fig.2.(a)). It means that the change of observing scale do not affect the observing result. In fact, when scale has changed, the image should be changed, too, the estimating of \(D\) should be based on the changed image (shown in Fig.2.(b)).

![Fig.1 Images with different scale](attachment:image1.png)

![Fig.2 Different scale lead to different results](attachment:image2.png)
3. A improved method for estimating fractal dimension

In this investigation, a improved method for estimating fractal dimension, which is based on the different resolution images, is proposed. In the method, according to the scale, a serial of images with different resolution are calculated with the original image, and then, the fractal dimension are estimated on the base.

Suppose the size of original image is $M \times N$. If $r$ and $r'(r>r)$ are the resolution of the original image and the changing scale image respectively, the relationship of two kinds of pixel is shown in Fig.3.

The value of a pixel is represented the average spectral energy of the corresponding area to the pixel on the ground. We can get a new image by calculating the average energy in area $r' \times r'$ with the value of the area $r \times r$. The pixel value of an image in which the resolution is $r'(r'>r)$ can be calculated by Eq.(9).

$$I^i = (r/r')^2 \sum_{i} I_i \times p(i)$$

(9)

where $I_i (i=1,2,\ldots,k)$ denotes $k$ values of original pixels which are in the area $r' \times r'$, $p(i)$ represents the area percent of area $r \times r$ for No. $i$ pixel in area $r' \times r'$. For example, if $r/r'=1.5$ in Fig.3, the area percents of pixel A, B, C, D are 50%, 25%, 100%, 50%, respectively. $(r/r')^2$ is a area ratio of original pixel and new pixel. It must be noted that the cover area on the ground of new image should be the same with the original image. It means that the size of the last column and row pixel on the new image are likely to smaller than $r' \times r'$, for example, pixels P, Q, R shown in Fig.3.

We use $F_r(x, y)$ denotes the image in which the scale is $r$. For the original image, let $r=1$. Eq.(8) represents the difference of two pixels in which the distance is a scale and in the image $F_r(x, y)$, the distance between two neighbor pixels is just a scale $r$(shown in Fig.2.(b)). So, the Eq.(8) can be modified as follow:

$$|F_r(x+\Delta x) - F_r(x)| = \frac{1}{3} \left( |F_r(x+1, y) - F_r(x, y)| + |F_r(x, y+1) - F_r(x, y)| + |F_r(x+1, y+1) - F_r(x, y)| \right)$$

(10)

Now, giving a reference $r$, with Eq.(9), we can get a new image $F_r(x, y)$; with Eq.(5), (6), (10), we can get a relationship about reference $H$; changing $r$, we can get a serial of the relationship; with a least-squares regression, we can estimate $H$, and thus D.

4. Results

In this investigation, two methods, the variance method and the improved method, are carried out on a SPOT image with five categories of objects: flourish field, bareness field, resident area, water area, and mountainous area. The representative examples of the five categories are shown in Fig.4. For each category, more than 100 regions, each of size $16 \times 16$ pixels, are then selected for the computation of fractal dimension with two kinds of methods. Here, the variable scale is $r=1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5$. A statistical calculation for each category has made to estimate the expectation $\mu$ and variance $\sigma$(shown in Tab.1). Suppose each category is followed normal distribution, the feature curve can be drawn with $\mu$ and $\sigma$(shown in Fig.5).
**5. Discussion**

Many natural textures are very similar to fractal surfaces, and the way the fractal model describing surface roughness correlates closely to human perception of texture. One of the most useful mathematical models for the random fractals found in nature has been fractional Brownian motion. We except that such a model can give us such a result. Observe Fig.4, we can find that water area is the smoothest and resident area is the most rough in all of five categories. Mountainous area is more rough than flourish field and bareness field, and two kinds of fields is very similar. From the Tab.1 and Fig.5, we can find the improved method, the paper proposed, can satisfy it.

During classification, we need to translate each sample into a feature space. We hope the samples of the same category can gather together and the samples of different category can separate widely so that we can distinguish different category through dividing different regions in the space. It needs a good mathematical model with which a smaller variance can be gotten. According to the experiment result, it can be found that the improved method is more helpful to classification than the variance method.

**6. Conclusion**

The improved method, proposed in this paper, can estimate, in a higher precision, the fractal dimension of natural phenomena in a remote sensing image. And the method is more helpful than the variance method. It means that the effect of improving variance method is successful.
7. References


