Modelling positional errors with isotropic random vector fields

João Casaca¹ and Ana Maria Fonseca¹
1 National Laboratory for Civil Engineering
Av. Brasil 101, 1700-066 Lisbon, Portugal
Tel.: + 351 21 844 3000; Fax: + 351 21 844 3026
jcasaca@lnec.pt; anafonseca@lnec.pt

Abstract

Analysis of spatially distributed random multidimensional phenomena, such as positional errors, requires some generalization from the usual concepts of Geostatistics: random vector fields, associating random vectors to space positions, must be used instead of random scalar fields; covariance matrix functions, must replace scalar covariance functions; matrix variograms, must replace scalar variograms; etc. In the case of isotropic vector random fields, which are well suited to model space distribution of positional errors, multidimensional generalization is easier, allowing the construction of a scalar pseudo-variogram which may be used as an efficient tool to partially estimate its autocovariance matrix functions. After a theoretical introduction, to present the concept of the scalar pseudo-variogram of an isotropic random vector field, the paper describes its application to the positional error vector of a geometrically corrected high resolution numeric image, acquired with the sensors of the Quickbird satellite.

Keywords: autocovariance function, isotropy, positional error vector, pseudo-variogram, vector random field.

1 Introduction

Recent airborne or spaceborne (Ikonos, Quickbird, etc.) sensors are able to acquire high geometric and radiometric resolution numeric images that may be used to produce numeric image maps, after being subject to the geometric corrections necessary to register them to topographical maps. The positional quality of the resulting image maps depends of the image resolution and the geometric correction procedures (number and position of the control points, orthorectification, etc.). The assessment of their positional quality is, therefore, essential to check their compliance to positional specifications or to compare the performance of different correction procedures.

The positional errors of point objects in numeric image maps are intuitively expected to be positively correlated: if two points are near, their positional errors are expected to be similar; if the points are far away, their positional errors are expected to be independent. Consequently, an isotropic random vector field provides a prior model to the distribution of positional error vectors that is adequate to the assessment of the numeric maps positional quality.

The evaluation of the distribution of the positional error vector is frequently carried out with a sampling strategy (unrestricted or stratified) that does not take into account the correlation between neighbour points and may introduce significant bias in the sample parameters (empirical mean and variance matrix). Under the isotropic model, the estimation of a pseudo-variogram from the sampled data provides an estimate of the autocovariance range that may be used as a lower bound to the size of sampling strata (Mailing, 1989), in order to produce inde-
dependent samples. Furthermore, the pseudo-variogram’s sill is an unbiased estimator of the trace of the unknown sample’s variance matrix.

2 Gaussian random vector fields
A two-dimensional Gaussian random vector field is an application:

\[ \Phi : D \subset \mathbb{R}^2 \to N(\mu, \Sigma) \]  

of a domain (D) of the Cartesian plane (\( \mathbb{R}^2 \)) into the family \( N(\mu, \Sigma) \) of two-dimensional Gaussian random vectors with mean vector \( \mu \) and variance matrix \( \Sigma \). The random vector field \( \Phi \) associates, to every point \( X \) of the domain \( D \), a Gaussian random vector \( U(X) \) defined by the mean vector \( \mu(X) \) and the variance matrix \( \Sigma(X) \).

The covariance matrix function of the vector random field \( \Phi \) is an application (K) of \( D \times D \) into the set of the symmetric matrices of order two, such that:

\[ K(X, Y) = E((E(U(X)))^T \Sigma(U(Y))) \]  

where \( E \) is the mathematical expectation operator. Whenever \( X = Y \), the covariance matrix function coincides with the variance matrix function:

\[ K(X, X) = \Sigma(X) \]  

A random vector field \( \Phi \) is said to be isotropic when its probabilistic structure is invariant under rigid body transformations (Stein, 1999). Necessary and sufficient conditions for a Gaussian random vector field to be isotropic in the domain \( D \) are: i) The mean vector function \( \mu(X) \) is null on \( D \); ii) The variance matrix function \( \Sigma(X) \) is constant \( \Sigma(X) = \Sigma \) on \( D \).

The matrix covariance function of a Gaussian isotropic random vector field, which is called an autocovariance function (Stein, 1999), depends exclusively of the distance (\( \rho \)) between the points:

\[ K(X, Y) = K(\rho) \]  

where:

\[ \rho = \|X - Y\| \]  

The previous definition implies that the autocovariance function at the origin (\( \rho = 0 \)) coincides with the (constant) variance matrix:

\[ K(0) = K(X, X) = \Sigma(X) \]  

Isotropic random vector fields \( \Phi \) which have an autocovariance function of the type:

\[ K(\rho) = \phi(\rho)\Sigma \]  

where \( \phi(\rho) \) is a continuous function of the distance (\( \rho \)) that verifies:

\[ \lim_{\rho \to 0} \phi(\rho) = 1 \land \lim_{\rho \to +\infty} \phi(\rho) = 0 \]
are called intrinsic autocorrelation isotropic random vector fields (Wackernagel, 2003). We will call the autocovariance functions of type (7) intrinsic autocovariance functions.

An example of an intrinsic autocovariance function is the exponential autocovariance:

\[ K(\rho) = \exp\left( -\frac{3\rho}{a} \right) \Sigma \]  

where the parameter \( a > 0 \), that controls the variation of the autocovariance function with the distance \( \rho \), is the effective range of the exponential autocovariance (Wackernagel, 2003). When the distance \( \rho \) is greater than the effective range \( a \), the exponential autocovariance is lesser than 0.05.

3 Theoretical and empirical pseudo-variograms

The properties of the distribution of quadratic statistics (Rao et al., 1988) may be used to derive unbiased estimators of the scale \((\varphi(\rho))\) and variance matrix \((\Sigma)\) of the intrinsic autocovariance function \((\varphi(\rho)\Sigma)\) of an intrinsic autocorrelation isotropic random vector field \(\Phi\).

For any pair of points \((X, Y)\) within the domain \(D\) of a random vector field \(\Phi\), the variance matrix of the random difference \((U(X) - U(Y))\) is:

\[ \text{Var}(U(X) - U(Y)) = \text{Var}(U(X)) + \text{Var}(U(Y)) - \text{Cov}(U(X), U(Y)) \]  

If the vector random field \(\Phi\) is Gaussian and isotropic, with intrinsic autocovariance function of the type (7), then the variance matrix may be expressed as:

\[ \text{Var}(U(X) - U(Y)) = 2(1 - \varphi(\rho))\Sigma \]  

Given a random n-sized sample \((U(X_1), ..., U(X_n))\) of the random vector field \(\Phi\), then the random matrices:

\[ G_{ij} = \frac{1}{2} (U(X_i) - U(X_j))(U(X_i) - U(X_j))^T \]  

are unbiased estimators of the matrix function:

\[ \Gamma(\rho_{ij}) = (1 - \varphi(\rho_{ij}))\Sigma \]  

where \(\rho_{ij} = \| X_i - X_j \|\). Furthermore, the random variables:

\[ g_{ij} = \frac{1}{2} (U(X_i) - U(X_j))^T (U(X_i) - U(X_j)) \]  

are unbiased estimators of the scalar function:

\[ \gamma(\rho_{ij}) = (1 - \varphi(\rho_{ij}))\text{Tr}(\Sigma) \]  

where \(\text{Tr}\) is the trace operator. The variances of the estimators \(g_{ij}\) are:
\[ \text{Var}(g_{ij}) = 2(1 - \varphi(\rho_{ij})) \text{Tr}(\Sigma) = (1 - \varphi(\rho_{ij}))(2\sigma_1^2 + 4\sigma_1^2 + 2\sigma_2^4) \]  

(16)

The matrix function \( \Gamma(\rho) \) (13) and the random matrices \( G_{ij} \) (12) may be regarded as multidimensional generalizations of the theoretical and empirical variograms of a scalar random field, so we will call them, respectively, the theoretical and the empirical multi-variograms of the random field \( \Phi \).

The scalar function \( \gamma(\rho) \) (15) and the random variables \( g_{ij} \) (14), though conceptually different, look like the theoretical and the empirical variograms of a scalar random field, so we will call them, respectively, the theoretical and the empirical pseudo-variograms of the random vector field \( \Phi \).

4 Estimation of the autocovariance

The observed values of the random matrices \( G_{ij} \) and the random variables \( g_{ij} \) may be used to estimate the autocovariance function \( \varphi(\rho)\Sigma \). In a first stage, a least squares fitting of the observed \( g_{ij} \) to the pseudo-variogram \( \gamma(\rho) \) provides the parameters of the autocovariance scale \( \varphi(\rho) \).

A second least squares fitting of the observed matrices \( G_{ij} \) and the computed covariance scales \( \varphi(\rho_{ij}) \) to the multi-variogram \( \Gamma(\rho) \) provides an estimate of the variance matrix \( \Sigma \).

In order to fit with least squares the observed \( g_{ij} \) to the pseudo-variogram \( \gamma(\rho) \), this one must be linearized. Presuming an exponential autocovariance (9), the application of Taylor’s formula to the pseudo-variogram leads to:

\[ \gamma(\rho \mid (a_0 + da, c_0 + dc)) = \gamma(\rho \mid (a_0, c_0)) + \left( \frac{\partial \gamma}{\partial a} \right)_{a_0} da + \left( \frac{\partial \gamma}{\partial c} \right)_{c_0} dc \]  

(17)

where \( a_0 \) is an approximation of the effective range \( a \), \( c_0 \) is an approximation of the sill \( c \) and where the partial derivatives are:

\[ \frac{\partial \gamma}{\partial a} = \frac{3c}{a} \exp\left(-\frac{3p}{a}\right) \]  

\[ \frac{\partial \gamma}{\partial c} = 1 - \exp\left(-\frac{3p}{a}\right) \]  

(18)

The empirical variance matrix \( S \) of the sample \( (U(X_1), ..., U(X_n)) \) provides a fair initial approximation of the pseudo-variogram’s sill \( c_0 = \text{Tr}(S) \). An initial approximation of the effective range \( a_0 \) may be computed with the observed \( g_{ij} \) and \( c_0 \).

The initial values \( (a_0, c_0) \) may be used as the seed of a Newton iterative process, built with relation (17), to get corrections \( da \) and \( dc \) to the initial values and, consequently, least squares estimates \( (a_0 + da) \) and \( (c_0 + dc) \) of the unknown effective range \( a \) and sill \( c \), respectively.

5 Example

A pan-sharpened multispectral image of Lisbon, acquired with the P and XS sensors of the Quickbird satellite, was geometrically corrected (registered to a large scale topographical map) with an affine transformation (6 parameters) of the software package Image Analyst. The geo-
metric correction procedure was supported by 15 control points. An independent group of 60 test points was reserved to control the positional accuracy of the corrected image. The distances between the test points were evenly spread between 20m and 3km.

The mean vector (M), the variance matrix (S) and the correlation matrix (R), computed from the positional error vectors (U(X₁), ..., U(X₆₀)) measured at the 60 test points (Morrison, 1990), are presented at Table 1.

<table>
<thead>
<tr>
<th>M (m)</th>
<th>S (m²)</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.414</td>
<td>2.066</td>
<td>−0.709</td>
</tr>
<tr>
<td>0.042</td>
<td>−0.709</td>
<td>0.588</td>
</tr>
</tbody>
</table>

Replacing the unknown theoretical variance matrix (Σ) by the empirical variance matrix (S) and using the quantile of probability 0.95 of the central chi-square distribution with two degrees of freedom, it’s possible to compute the parameters of the 0.95 probability error ellipse for the positional error vector of the corrected image. For a 0.95 probability level, the major and the minor semi-axis of the error ellipse are a = 3.76m and b = 1.33m, respectively. The ellipse presents a significant flattening, meaning that the distribution of the positional error is not homogeneous.

The positional error vectors lie within the error ellipse of the image with a probability 0.95 and, consequently, they are contained within a circle (the error circle) with a radius equal to the major semi-axis of the ellipse (3.76m) with a probability greater than 0.95. The error circle may be used as a simple criterion to evaluate the positional accuracy of the geometrically corrected numeric image or as a tolerance region.

A theoretical pseudo-variogram γ(ρ), with effective range a = 932m and sill c = 2.84m², was adjusted to the 1770 values of the variable gᵢⱼ, that resulted from the dissimilarities (14) between the positional error vectors measured at the 60 test points. The estimated standard deviation of the unit weight, which measures the quality of the least squares adjustment, was 3.22m.

The estimated effective range (a ≈ 1km) implies a significant covariance between positional errors at different points of the geometrically corrected image. The estimated sill (c ≈ 2.84m²), which is also an estimate of Tr(Σ), agrees with Tr(S) = 2.65m², taking into account that the presence of a significant covariance influences Tr(S) to underestimate Tr(Σ).

6 Conclusion
The conventional criteria used to assess the positional accuracy of numeric image maps that consider independently errors in Easting and errors in Northing (Mailing, 1989), are quite inadequate since they ignore: i) Correlation between Easting and Northing components of the error vectors; ii) Covariance between the error vectors at different positions.

The isotropic autocovariance function provides an intuitive model to the distribution of positional errors, since they are expected to be similar at neighbour points, that is advantageous from the operational point of view. The range of the autocovariance function is a straightforward guideline to organize a stratified sampling strategy to positional quality control of numeric image maps.
References
Stein, M., 1999, Interpolation of Spatial Data, Berlin: Springer.